Strength of Anisotropic Materials under Combined Stresses

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A criterion is presented for the prediction of failure of anisotropic materials under combined loading. Although tensorial in nature, the criterion is represented in a matrix form for ease of visualization. The approach is developed in detail for orthotropic materials under plane stress, but is applicable to general three-dimensional anisotropic materials. Two concepts are introduced: the interaction factor, which is a measure of strength anisotropy; and the principal strength axes, which are related to the directionality of the material strength properties. The criterion provides description of strength phenomena peculiar to fiber composite materials, but it is also applicable to homogeneous anisotropic metals.

Nomenclature

$h(\gamma), g(\gamma), f(\gamma)$	= functions of interaction factor γ used to express failure criterion with respect to axes of orthotropy
$[R]$	= fourth-order strength tensor represented in matrix format
T	= simple failure stress, strength under simple load condition
α, β, γ	= Interaction factors, defined by the relations $\gamma = 3T_{xy}^2/T_xT_y$, $\beta = 3T_{zx}^2/T_zT_x$, $\alpha = 3T_{zy}^2/T_zT_y$
$\{\sigma\}$	= second-order stress tensor represented in vector format
σ, τ	= normal stress, shear stress
δ	= angle related to the oblique coordinate system
θ	= rotation angle
μ	= design factor of safety (the minimum of μ_1 and μ_2)
$[\Lambda_\delta]$	= transformation between oblique coordinate system and orthogonal coordinate system represented in matrix format
$[\Gamma_\theta]$	= coordinate rotation operator, represented in matrix format
Coordinates	
x', y', z'	= arbitrary orthogonal axes
x, y, z	= axes of orthotropy
x_1, y_1, z_1	= principal strength axes
x_0, y_0	= intermediate orthogonal axes, used for convenience
Superscripts	
T	= vector or matrix transpose
Subscripts	
x', y', z'	= arbitrary orthogonal axes
x, y, z	= axes of orthotropy
x_1, y_1, z_1	= principal strength axes
1	= principal strength axes
T, C	= tensile, compressive quantity
p	= plane stress representation

I. Introduction

Phenomenological Failure Theories for Composite Materials

A PHENOMENOLOGICAL failure theory is actually a design criterion, by means of which the strength of materials under combined stresses can be predicted on the basis of strength data obtained from relatively simple uniaxial and shear tests. These criteria are intended to predict occurrence, rather than mode, of failure. As Refs. 1–27 indicate, a large number of such criteria have been proposed for anisotropic materials, whereas the von Mises criterion for yielding of ductile materials and the Mohr's criterion for fracture of brittle materials are examples of phenomenological failure theories applied to isotropic materials.

When predicting failure of laminated composite materials, the existing failure criteria are currently applied in one of two ways: a) The laminate is assumed to consist of homogeneous orthotropic plies. It is analyzed, ply for ply, using an interaction formula as the failure criterion. Failure is presumed to occur when the criterion is not met in any ply or group of plies. b) The laminate is assumed homogeneous and anisotropic (not necessarily orthotropic). The criterion is applied to the laminate as a whole. The first procedure requires basic strength data only for each *type* of ply in the laminate, but requires involved computation of the stress distribution through the layers for each combined load condition and laminate orientation. The second procedure, on the other hand, does not require computation of this stress distribution, since it is based on strength data obtained for the whole laminate, but it does require basic strength data for each lamination pattern being considered. Upon examination of these points, a third, or "hybrid" procedure suggests itself: c) For a given laminate, use the plywise procedure a to determine strengths under simple uniaxial and shear load conditions. Having obtained these strengths for the laminate, the whole-laminate procedure b is then used to determine strengths under combined load conditions. Using this third procedure, only basic ply strength data would be required, and it would be necessary to determine the complete stress distribution for only a few basic load systems. With this reduced number of cases, more sophisticated computer programs could be used to obtain more detailed descriptions of the basic stress fields and, presumably, more accurate determinations of basic laminate strengths.

In well-designed laminated composite structures, the plies are oriented so that failure is precipitated by fiber breakage rather than matrix separation. In this case, the failure criterion is actually a "fracture" criterion. On the other hand, when homogeneous anisotropic materials such as rolled titanium are studied, a method of type b must necessarily be applied. In these cases, the failure criteria may actually be "yield" criteria.

The main obstacle to implementation of the third scheme c is the lack of usable failure criteria for homogeneous anisotropic materials. It will be shown in the next section that the interaction formulas commonly applied when using the ply-by-ply procedure a will lead to paradoxes and unanswered questions when applied directly to the entire laminate. This paper presents a failure criterion which is applicable to whole laminates, yet avoids the paradoxes inherent to existing criteria.

Notes on the Invariance of Failure Criteria

The majority of failure criteria currently used in the field of composites are quadratic in form. These criteria are typified by the Hill-type interaction equation:

$$(\sigma_x/T_x)^2 - (\sigma_x\sigma_y/T_xT_y) + (\sigma_y/T_y)^2 + (\tau_{xy}/T_{xy})^2 = 1 \qquad (1)$$

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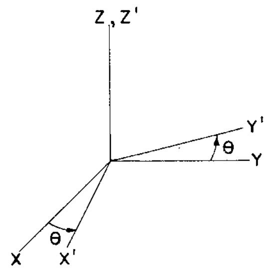

Fig. 1 Coordinate systems x, y and x', y'.

where T_x and T_y are uniaxial tensile (compressive) strengths, T_{xy} is the shear strength, and the stresses $\sigma_x, \sigma_y, \tau_{xy}$ are referred to the axes of orthotropy x, y. When it is necessary to evaluate a stress system $\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}$ referred to a rotated coordinate system x', y' (Fig. 1), it is commonly postulated (Hill[2,3] and Tsai et al.[7]) that the criterion (1) is still applicable if these stresses are transformed to the axes of orthotropy, i.e.,

$$\{\sigma\} = [\Gamma_\theta]\{\sigma'\} \tag{2}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}; \qquad \{\sigma'\} = \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{Bmatrix}$$

$$[\Gamma_\theta] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -\sin 2\theta \\ \sin^2\theta & \cos^2\theta & \sin 2\theta \\ \frac{1}{2}\sin 2\theta & -\frac{1}{2}\sin 2\theta & \cos 2\theta \end{bmatrix}$$

The quadratic form (1) can also be expressed in matrix form

$$\{\sigma\}^T[R]\{\sigma\} = 1 \tag{3}$$

where

$$[R] = \begin{bmatrix} (1/T_x)^2 & -(1/2T_xT_y) & 0 \\ -(1/2T_xT_y) & (1/T_y)^2 & 0 \\ 0 & 0 & (1/T_{xy})^2 \end{bmatrix}$$

By the postulate, and using Eq. (2):

$$\{\sigma\}^T[R]\{\sigma\} = \{\sigma'\}^T[\Gamma_\theta]^T[R][\Gamma_\theta]\{\sigma'\} = 1$$

or

$$\{\sigma'\}^T[R']\{\sigma'\} = 1 \tag{4}$$

where

$$[R'] = [\Gamma_\theta]^T[R][\Gamma_\theta] \tag{5}$$

In this matrix formulation of the Hill-type criterion, the quantities $[R]$ and $[R']$ can be called the strength matrices defined in the two coordinate systems. Since the elements of the strength matrix $[R]$ are not functions of the elements of the column vector $\{\sigma\}$, the postulate leading to (5) is equivalent to requiring that the interaction Eq. (1) be a tensorial statement. Thus, *as a consequence* of this postulate, the strength "matrix" $[R]$ can be transformed to any coordinate system, in a manner analogous to that in which the elastic constants $[C]$ of an anisotropic material are transformed. In fact, any criterion, quadratic or otherwise, subject to this postulate is inherently tensorial.

The authors have not seen this fact explicitly pointed out anywhere in the literature, so it is appropriate to state it herein, for later reference.

Although the quadratic form (1) may seem more convenient than the matrix (tensor) form (3), the latter is much more appropriate for demonstrating the paradoxes which arise in the applications of this type of failure criterion.

A Paradox of the Hill-Type Criterion

The Hill-type criterion (1) is generally applicable only to individual plies of the laminate; the axes x and y in (1) are strictly defined as the axes of orthotropy of the ply, and are coincident with the fiber direction and its perpendicular. The fact that this criterion cannot be applied directly [as in method b, described earlier] to even the simplest multidirectional laminate will be demonstrated in the following discussion.

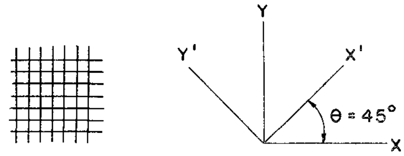

Fig. 2 Balanced 0°–90° laminate.

Consider a balanced 0°–90° laminate consisting of an equal number of identical plies in each direction, as sketched in Fig. 2. There are two sets of axes of orthotropy in this laminate: the axes x, y oriented along the fiber directions and the axes x', y' oriented along the $\pm 45°$ directions. Let us first consider the former set x, y oriented along the fiber directions. Since the plies of the 0°–90° laminate are identical, the laminate strengths T_x and T_y are equal ($T_x = T_y = T$). When the criterion (1) is applied directly to this laminate, the strength matrix $[R]$ can be written

$$[R] = \begin{bmatrix} 1/T^2 & -(1/2T^2) & 0 \\ -(1/2T^2) & 1/T^2 & 0 \\ 0 & 0 & 1/T_{xy}^2 \end{bmatrix} \tag{6}$$

Correspondingly, setting $\theta = 45°$ in (2) and using (5), the transformed strength matrix $[R']$ in the rotated coordinate system x', y' can be written

$$[R']_1 = \begin{bmatrix} \frac{1}{4}(1/T^2 + 1/T_{xy}^2) & -\frac{1}{4}(1/T_{xy}^2 - 1/T^2) & 0 \\ -\frac{1}{4}(1/T_{xy}^2 - 1/T^2) & \frac{1}{4}(1/T^2 + 1/T_{xy}^2) & 0 \\ 0 & 0 & 3/T^2 \end{bmatrix} \tag{7}$$

On the other hand, since the axes x', y' are also axes of orthotropy the strength matrix $[R']_2$ for the $\pm 45°$ balanced laminate could also be stated directly from (1), as

$$[R']_2 = \begin{bmatrix} 1/T'^2 & -(1/2T'^2) & 0 \\ -(1/2T'^2) & 1/T'^2 & 0 \\ 0 & 0 & 1/T_{x'y'}^2 \end{bmatrix} \tag{8}$$

where T' and $T_{x'y'}$ are the strengths referred to the axes x', y'. If the Hill-type criterion is to produce consistent predictions for the laminate, it must follow, upon comparing (7) with (8) that

$$1/T'^2 = \tfrac{1}{4}(1/T^2 + 1/T_{xy}^2), \quad 1/T_{x'y'}^2 = 3/T^2 \tag{9}$$

in which case $[R']$ could be restated in terms of T' and $T_{x'y'}$

$$[R']_1 = \begin{bmatrix} 1/T'^2 & -(1/T'^2 - 1/6T_{x'y'}^2) & 0 \\ -(1/T'^2 - 1/6T_{x'y'}^2) & 1/T'^2 & 0 \\ 0 & 0 & 1/T_{x'y'}^2 \end{bmatrix} \tag{10}$$

When the two forms of the failure criterion for the $\pm 45°$ laminate are plotted in the $\tau_{x'y'} = 0$ plane, Fig. 3, it becomes evident that the interaction curve obtained by transformation, using $[R']_1$, does not generally coincide with that obtained from the direct statement, using $[R']_2$. This occurs because the off-diagonal terms of these two strength tensors are never equal unless $T_{xy} = T/(3)^{1/2}$, i.e., unless the material is isotropic.

It is evident from this paradoxical result that the Hill-type criterion cannot be applied directly to the whole laminate, as described in method b. However it is still effective as an interaction formula when referred to the ply axes of orthotropy x, y and when used as indicated in method a.

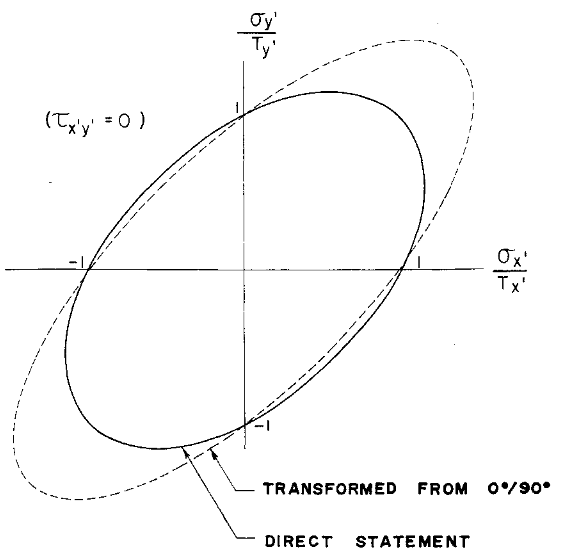

Fig. 3 Comparison of Hill-type failure surfaces for a ±45° balanced laminate.

In order for a whole-laminate criterion to avoid the paradox just described, it is necessary to provide a more specific definition of the reference axes that must be used. Moreover, this definition must permit determination of these reference axes, even when the material is not orthotropic. A rational method for defining these axes will be discussed in the following sections.

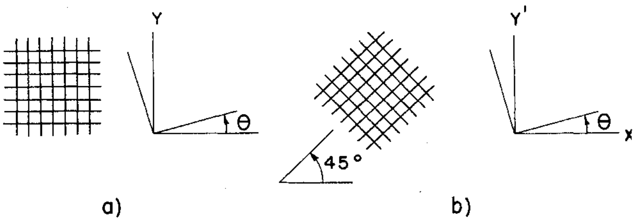

Fig. 4 Reinforcement of an isotropic matrix: a) reinforced in 0°–90° directions; b) reinforced in ±45° directions.

The Interaction Factor

Consider a parameter γ defined by the relation

$$\gamma = 3T_{xy}^2/T_x T_y \tag{11}$$

where T_x, T_y, and T_{xy} are basic strengths determined relative to the x, y coordinate system. When a material is isotropic and obeys the von Mises failure criterion ($T_x = T_y = (3)^{1/2}T_{xy}$), this parameter is numerically equal to unity. When this isotropic material is equally reinforced by strong fibers oriented along the x and y coordinate directions as shown in Fig. 4a, the strengths $T_x = T_y$ are increased, while relatively little change occurs in the shear strength T_{xy}. In this case the parameter γ becomes less than unity. Moreover, the tensile strengths T_x and T_y are relative maxima with respect to the rotation angle θ, while the shear strength T_{xy} is a relative minimum. If, on the other hand, the isotropic matrix is reinforced equally by strong fibers oriented ±45°, as shown in Fig. 4b, the shear strength $T_{x'y'}$ is increased more than the tensile strengths $T_{x'} = T_{y'}$. In this case, the shear strength $T_{x'y'}$ is a relative maximum with respect to rotation θ, whereas $T_{x'}$ is a relative minimum; and the parameter γ, defined as $\gamma = 3T_{x'y'}^2/T_{x'}T_{y'}$ is greater than unity. The parameter γ (or γ'), hereafter designated the "interaction factor," is obviously a function of the orientation of the axes for a given material: it achieves a relative minimum less than unity when the material is reinforced as shown in Fig. 4a, and achieves a relative maximum greater than unity when the material is reinforced as shown in Fig. 4b. When these two materials are the same, the value of this parameter provides a basis for distinguishing between the x, y and x', y' axes of orthotropy shown in Fig. 1.

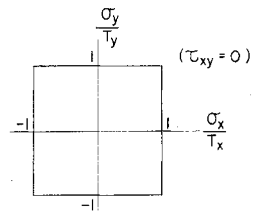

Fig. 5 Failure surface for fabric-like material.

Principal Axes of Strength

It is now possible to define the principal strength axes of a material as those axes for which the parameter γ is a relative minimum with respect to rotation. By this definition, the axes x, y of Fig. 1 are the principal strength axes.

The significance of the parameter γ is further demonstrated in the case of a fabric-like material, made up of strong fibers oriented along the x and y coordinate directions but with very weak matrix material. For purpose of discussion, let us designate this an "ideal noninteracting" material. For this material, the shear strength T_{xy} approaches zero, whereas the strengths T_x and T_y come to depend mostly upon the strengths of the reinforcing fibers. The interaction factor γ, defined by (11), approaches zero. The corresponding failure surface in the $\tau_{xy} = 0$ plane approaches the square shape shown in Fig. 5. Predictions for the ideal noninteracting material are identical to those attained from netting analysis (shear effects ignored). When Fig. 5 is compared against Fig. 3, two things should become evident: First, the Hill-type criterion does not provide for a transition between the von Mises ellipse (for $\gamma = 1$) and the rectangle ($\gamma = 0$). Not only must some measure of interaction, such as γ, be used to choose the reference axes, but it must also be incorporated into the mathematical statement of the criterion in order to provide for this transition. Second, no single continuous closed surface of the form (1) can be made to perform the transition between the ellipse for $\gamma = 1$ and the rectangle for $\gamma = 0$. The criterion must be stated in the form of several equations, representing several intersecting surfaces in the $\{\sigma_x \sigma_y \tau_{xy}\}$ space.

In the next section, a failure criterion is proposed which possesses these features.

II. The Proposed Failure Criterion

General Case

The discussion of the previous sections can be generalized to the three dimensional case. First, define the interaction factors α_1, β_1, γ_1 by the relations:

$$\alpha_1 = \frac{3T_{y_1z_1}^2}{T_{y_1}T_{z_1}}; \quad \beta_1 = \frac{3T_{z_1x_1}^2}{T_{z_1}T_{x_1}}; \quad \gamma_1 = \frac{3T_{x_1y_1}^2}{T_{x_1}T_{y_1}} \tag{12}$$

where x_1, y_1, z_1 are principal strength axes and T_{x_1}, T_{y_1}, T_{z_1}, $T_{x_1y_1}$, $T_{x_1z_1}$, and $T_{z_1y_1}$ are the corresponding uniaxial (tensile or compressive) and shear strengths of the material. In accord with the preceding discussion, the failure criterion is then stated in three quadratic expressions of the form

$$\{\sigma_1\}^T[R_1^{(i)}]\{\sigma_1\} = 1, \quad i = 1, 2, 3 \tag{13}$$

where the applied stress "vector" $\{\sigma_1\}$ is defined by

$$\{\sigma_1\} = \{\sigma_{x_1}\sigma_{y_1}\sigma_{z_1}\tau_{y_1z_1}\tau_{z_1x_1}\tau_{x_1y_1}\}$$

and the strength "matrices" are postulated as follows:

$$[R_1^{(i)}] = \left[\begin{array}{c|c} A_1^{(i)} & 0 \\ \hline 0 & B_1 \end{array}\right] \tag{14}$$

The submatrices $[A_1^{(i)}]$ and $[B_1]$ are defined by

$$[A_1^{(1)}] = \begin{bmatrix} 1/T_{x_1}^2 & -\gamma_1/2T_{y_1}^2 & -\beta_1/2T_{z_1}^2 \\ & \gamma_1/T_{y_1}^2 & -1/2T_{x_1}^2 \\ \text{symm} & & \beta_1/T_{z_1}^2 \end{bmatrix}$$

$$[A_1^{(2)}] = \begin{bmatrix} \gamma_1/T_{x_1}^2 & \gamma_1/2T_{x_1}^2 & -(1/2T_{y_1}^2) \\ & 1/T_{y_1}^2 & -(\alpha_1/2T_{z_1}^2) \\ \text{symm} & & \alpha_1/T_{z_1}^2 \end{bmatrix}$$

$$[A_1^{(3)}] = \begin{bmatrix} \beta_1/T_{x_1}^2 & -(1/2T_{z_1}^2) & -(\beta_1/2T_{x_1}^2) \\ & \alpha_1/T_{y_1}^2 & -(\alpha_1/2T_{y_1}^2) \\ \text{symm} & & 1/T_{z_1}^2 \end{bmatrix} \tag{15}$$

$$[B_1] = \begin{bmatrix} 1/T_{y_1z_1}^2 & 0 & 0 \\ 0 & 1/T_{z_1x_1}^2 & 0 \\ 0 & 0 & 1/T_{x_1y_1}^2 \end{bmatrix}$$

Note that this criterion possesses the features outlined previously: The three equations represent three intersecting surfaces in the stress space; the interaction factors α_1, β_1, γ_1 are used in the choice of principal strength axes; and these interaction factors are included into the mathematical form of the criterion.

The values of the strengths T_{x_1}, T_{y_1}, and T_{z_1} in (15) are chosen in accord with the signs of the normal stresses. For example, if only σ_{x_1} and σ_{y_1} are different from zero, and if the material has tensile strengths T_{x_1T}, T_{y_1T}, and compressive strengths T_{x_1C}, T_{y_1C}, the values of T_x and T_y in Eqs. (14) and (15) are chosen by the following rule

$$\sigma_{x_1} \geq 0: T_{x_1} = T_{x_1T}, \quad \sigma_{x_1} < 0: T_{x_1} = T_{x_1C},$$
$$\sigma_{y_1} \geq 0: T_{y_1} = T_{y_1T}, \quad \sigma_{y_1} < 0: T_{y_1} = T_{y_1C} \tag{16}$$

When the material is isotropic, the interaction factors are unity and the three expressions (13) each reduce identically to the von Mises criterion for isotropic materials.

Under a hydrostatic state of stress ($\sigma_{x_1} = \sigma_{y_1} = \sigma_{z_1} = \sigma$ and $\tau_{y_1z_1} = \tau_{z_1x_1} = \tau_{x_1y_1} = 0$), this failure criterion is never met, implying that no material failure occurs even though distortions could exist under such a state of stress.

Plane Stress Case

Application of this criterion will be demonstrated for a material in a state of plane stress. In addition to its more wide-spread applicability, the plane stress theory is more easily supplemented by practical examples. In this discussion, strength properties and stresses in the z_1 coordinate will be ignored, and the failure criterion can be conveniently represented in a reduced form analogous to (3)

$$\{\sigma_1\}_p^T[R_1^{(i)}]_p\{\sigma_1\}_p = 1; \quad i = 1, 2 \tag{17}$$

where

$$\{\sigma_1\}_p = \{\sigma_{x_1}\sigma_{y_1}\tau_{x_1y_1}\}$$

and

$$[R_1^{(1)}]_p = \begin{bmatrix} 1/T_{x_1}^2 & -(\gamma_1/2T_{y_1}^2) & 0 \\ & \gamma_1/T_{y_1}^2 & 0 \\ \text{symm} & & 1/T_{x_1y_1}^2 \end{bmatrix}$$

$$[R_1^{(2)}]_p = \begin{bmatrix} \gamma_1/T_{x_1}^2 & -(\gamma_1/2T_{x_1}^2) & 0 \\ & 1/T_{y_1}^2 & 0 \\ \text{symm} & & 1/T_{x_1y_1}^2 \end{bmatrix} \tag{18}$$

$$(\gamma_1 = 3T_{x_1y_1}^2/T_{x_1}T_{y_1})$$

In explicit form, the failure criterion (17) can be written

$$\left(\frac{\sigma_{x_1}}{T_{x_1}}\right)^2 - \gamma_1\left(\frac{T_{x_1}}{T_{y_1}}\right)\left(\frac{\sigma_{x_1}}{T_{x_1}}\right)\left(\frac{\sigma_{y_1}}{T_{y_1}}\right) + \gamma_1\left(\frac{\sigma_{y_1}}{T_{y_1}}\right)^2 + \left(\frac{\tau_{x_1y_1}}{T_{x_1y_1}}\right)^2 = 1$$
$$\gamma_1\left(\frac{\sigma_{x_1}}{T_{x_1}}\right)^2 - \gamma_1\left(\frac{T_{y_1}}{T_{x_1}}\right)\left(\frac{\sigma_{x_1}}{T_{x_1}^2}\right)\left(\frac{\sigma_{y_1}}{T_{y_1}}\right) + \left(\frac{\sigma_{y_1}}{T_{y_1}}\right)^2 + \left(\frac{\tau_{x_1y_1}}{T_{x_1y_1}}\right)^2 = 1 \tag{19}$$

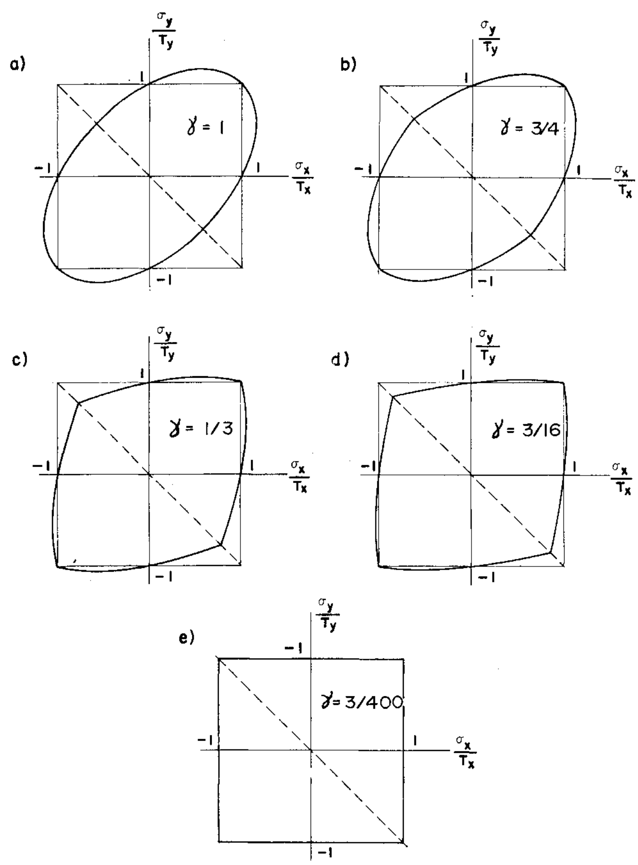

Fig. 6 Failure curves for orthotropic materials with $T_{x1} = T_{y1} = T$, $T_{x1y1} \leq T/(3)^{1/2}$: a) $T_{x1y1} = T/(3)^{1/2}$, $\gamma_1 = 1$; b) $T_{x1y1} = T/2$, $\gamma_1 = 3/4$; c) $T_{x1y1} = T/3$, $\gamma_1 = 1/3$; d) $T_{x1y1} = T/4$, $\gamma_1 = 3/16$; e) $T_{x1y1} = T/20$, $\gamma_1 = 3/400$.

The corresponding $\tau_{x_1y_1} = 0$ failure surfaces are plotted in Fig. 6 for the case $T_x = T_y = T$, $T_{xy} \leq T/(3)^{1/2}$. When the material is isotropic, the interaction factor γ_1 is equal to unity, and both equations define the "von Mises ellipse" of Fig. 6a. When the shear strength is reduced, the interaction factor is less than unity and Eqs. (19) yield two distinct ellipses. Figures 6b, 6c, 6d, and 6e illustrate the progressive change of the failure surface in the $\tau_{x_1y_1} = 0$ plane, due to the interaction of these ellipses. For the fabric-like material ($\gamma_1 = 0$), the failure surface becomes a square.

The dashed line in the second quadrant of Fig. 5 corresponds to a pure shear stress $\tau_{x'y'}$ in a coordinate system rotated 45° with respect to the principal strength axes. Note that as the shear strength $T_{x_1y_1}$ decreases (i.e., as γ_1 decreases), the shear strength $T_{x'y'}$ in this rotated system actually increases. Although this behavior appears paradoxical, it is consistent with experience. For example, consider the 0°/90° laminate, with strengths $T_{x_1} = T_{y_1} = T$. Figure 6 indicates that the shear strength $T_{x'y'}$ in the ±45° laminate will always fall in the range $T \leq T_{x'y'} \leq T/(3)^{1/2}$, where T is the prediction obtained from netting analysis ($\gamma_1 = 0$) and $T/(3)^{1/2}$ is the prediction for an isotropic material ($\gamma_1 = 1$, Fig. 6a). Available test results for boron/epoxy laminates[24] support this conclusion.

Oblique Principal Strength Axes

In most practical composites, the lamination pattern is known, and it would appear that the principal strength axes can be determined from knowledge of this pattern, rather than from actual tests. Since most currently-used patterns produce orthotropic laminates, it is natural to infer that the principal strength axes should always be orthogonal and coincident with certain intuitively determined principal axes of orthotropy. It will be shown, however, that the principal strength axes to not necessarily coincide with the axes of orthotropy, nor do they even need to be orthogonal.

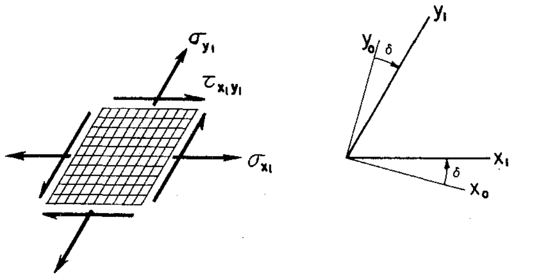

Fig. 7 "Ideal noninteracting material" with oblique principal strength axes.

An ideal noninteracting material ($\gamma_1 = 0$) with oblique principal axes could be visualized as a "fabric" of weakly connected fibers, oriented in nonorthogonal directions x_1, y_1, as shown in Fig. 7. This fabric is obviously capable of supporting the tensile or compressive stresses σ_{x_1}, σ_{y_1} defined by Fig. 7, but not the shear stress $\tau_{x_1y_1}$. Note that the stresses defined in this figure are not the conventional normal and shear stresses, but are force-related quantities defined with respect to the oblique coordinate system. It is convenient to transform these oblique stresses to conventional orthogonal stresses, using the transformation

$$\{\sigma_1\} = [\Lambda_\delta]\{\sigma_0\} \tag{20}$$

where $\{\sigma_1\}$ are the oblique stress components, $\{\sigma_0\}$ are the stress components referred to the intermediate orthogonal coordinate system x_0, y_0 of Fig. 8, and the transformation matrix $[\Lambda_\delta]$ is a function of the angle δ

$$[\Lambda_\delta] = \begin{bmatrix} \dfrac{\sec 2\delta + 1}{2} & \dfrac{\sec 2\delta - 1}{2} & -\tan 2\delta \\ \dfrac{\sec 2\delta - 1}{2} & \dfrac{\sec 2\delta + 1}{2} & -\tan 2\delta \\ \dfrac{-\tan 2\delta}{2} & \dfrac{-\tan 2\delta}{2} & \sec 2\delta \end{bmatrix}$$

Now, using the transformation $[\Gamma_\theta]$ given in (2), it is possible to pass from the intermediate orthogonal system x_0, y_0 to any other orthogonal set x', y' using

$$\{\sigma_0\} = [\Gamma_\theta]\{\sigma'\} \tag{21}$$

where θ is the angle of rotation.

Thus, introducing Eqs. (20) and (21) into (17), the general statement of the failure criterion is obtained, referred to any set of orthogonal axes x', y'

$$\{\sigma'\}^T[R'^{(i)}]\{\sigma'\} = 1; \quad i = 1, 2$$

with

$$[R'^{(i)}] = [\Gamma_\theta]^T[\Lambda_\delta]^T[R_1^{(i)}][\Lambda_\delta][\Gamma_\theta] \tag{22}$$

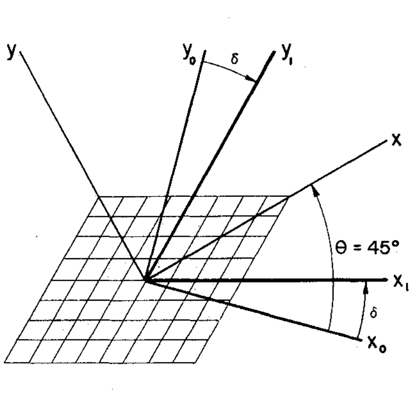

Fig. 8 Transformation from oblique principal axes x_1, y_1 to orthogonal axes x, y.

Failure Theory for Orthotropic Materials

In principle, at least, the failure of the laminate can be characterized completely if the principal strength axes x_1, y_1 are known, and the strengths T_{x_1}, T_{y_1}, $T_{x_1y_1}$ are determined. Practical application of (22) would be very difficult for the most general anisotropic materials, since the principal strength axes cannot be readily determined without extensive testing. When the material is orthotropic, however, it is possible to determine at least one set of principal axes if the strengths relative to any set of axes of orthotropy are known.

Using Figs. 7 and 8 as guides, it can be seen that the material is orthotropic in two circumstances

a) The principal strength axes are orthogonal ($\delta = 0$). In this case, the principal strength axes x_1, y_1 are identical to the axes of orthotropy x, y, and the strength tensors $[R_1^{(i)}]_p$ can be applied in the form (18) directly. When expressed in this coordinate system the interaction factor will not exceed unity ($\gamma_1 = \gamma \leq 1$).

b) The strengths T_{x_1}, T_{y_1} are equal. In this case, the principal strength axes are not necessarily orthogonal, but the axes located $\theta = 45°$ from the intermediate system x_0, y_0 are coincident with the axes of orthotropy x, y. From the strengths T_x, T_y, T_{xy}, determined in this coordinate system, the principal failure stresses T_{x_1}, T_{y_1}, $T_{x_1y_1}$ and angle δ can be calculated using (22), with $\theta = 45°$:

$$\frac{1}{T_{x_1}^2} = \frac{1}{T_{y_1}^2} = \frac{f(\gamma)}{T_xT_y}; \quad \frac{1}{T_{x_1y_1}^2} = \frac{4}{T_xT_y} - \frac{f(\gamma)}{T_xT_y} \tag{23}$$
$$\sin 2\delta = (T_x - T_y)/(T_x + T_y)$$

where

$$\gamma = 3T_{xy}^2/T_xT_y$$
$$f(\gamma) = \tfrac{1}{10}\{-(3/\gamma + 4) + [(3/\gamma + 4)^2 + 240/\gamma]^{1/2}\} \tag{24}$$

It is more convenient, in this case, to restate the strength tensors in the x, y coordinate system using the strengths T_x, T_y, T_{xy} directly

$$[R^{(i)}]_p = \begin{bmatrix} 1/T_x^2 & -h(\gamma)/T_xT_y & \pm g(\gamma)/T_xT_{xy} \\ & 1/T_y^2 & \pm g(\gamma)/T_yT_{xy} \\ \text{symm.} & & 1/T_{xy}^2 \end{bmatrix}; \quad i = 1, 2 \tag{25}$$

where the upper sign is used for $i = 2$, and where

$$h(\gamma) = 1 - \tfrac{1}{2}f(\gamma)$$
$$g(\gamma) = (\gamma/12)^{1/2}[3f(\gamma)/(f(\gamma) - 4) + 1]f(\gamma) \tag{26}$$

The functions $h(\gamma)$ and $g(\gamma)$ are represented in Fig. 9 for convenience. It can be shown, using Eqs. (23) and (24), that the condition

$$\gamma_1 = 3T_{x_1y_1}^2/T_{x_1}T_{y_1} \leq 1$$

implies that

$$\gamma = 3T_{xy}^2/T_xT_y > 1$$

Thus, for case b, the interaction factor γ will never be less than unity. Figure 10 illustrates the progression of the failure surface for an orthotropic material on the plane $\tau_{xy} = 0$. Note the changes of shape which take place as the interaction factor passes successively from zero (ideal noninteracting material: $T_{xy} = 0$), to values smaller than unity (materials relatively weak in shear: $3T_{xy}^2 < T_xT_y$), through unity (isotropic materials) and finally to values greater than unity (materials relatively strong in shear: $3T_{xy}^2 > T_xT_y$).

III. Applications to Orthotropic Materials Under Plane Stress

Procedure

Practical application of this theory to orthotropic materials can be summarized as follows: a) Establish simple failure stresses T_{x_T}, T_{y_T} in tension, T_{xc}, T_{yc} in compression, and T_{xy} in shear,

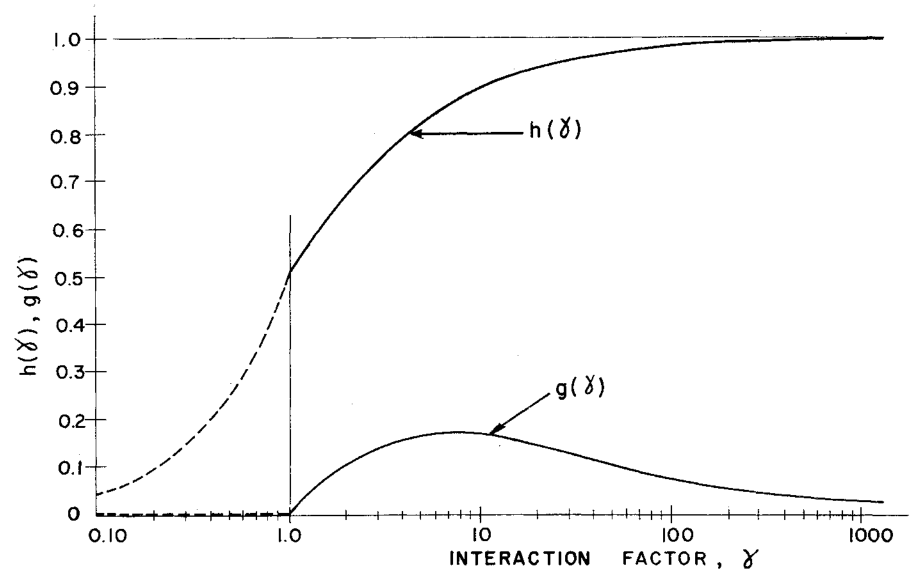

Fig. 9 The functions $h(\gamma)$ and $g(\gamma)$. The interaction factor γ is defined with respect to axes of orthotropy x, y.

referred to axes of orthotropy x, y. b) Determine the interaction factor $\gamma = 3T_{xy}^2/T_xT_y$, based upon the signs of the imposed stresses σ_x, σ_y [see expressions (16)]. c) Choose the appropriate strength matrices, using (18) when $\gamma \leq 1$ and (25) when $\gamma > 1$. These matrices are identical for isotropic materials ($\gamma = 1$). d) Compute the factor of safety for the given applied stresses.

Factor of Safety

If the applied stress condition is represented by the "vector" $\{\sigma_x\sigma_y\tau_{xy}\}$ in the stress space, the factor of safety μ is defined as that scalar factor which expands (or contracts) this vector so that $\{\mu\sigma_x\mu\sigma_y\mu\tau_{xy}\}$ terminates on the failure surface. In accord with this geometrical definition, when $\gamma \leq 1$, the factor of safety is the smaller of the quantities μ_1 and μ_2 defined below, using the strength matrices (18)

$$\frac{1}{\mu_1^2} = \left(\frac{\sigma_x}{T_x}\right)^2 + \gamma\left\{\left(\frac{\sigma_y}{T_y}\right)^2 - \left(\frac{T_x}{T_y}\right)\left(\frac{\sigma_x}{T_x}\right)\left(\frac{\sigma_y}{T_y}\right)\right\} + \left(\frac{\tau_{xy}}{T_{xy}}\right)^2 \quad (27a)$$

$$\frac{1}{\mu_2^2} = \left(\frac{\sigma_y}{T_y}\right)^2 + \gamma\left\{\left(\frac{\sigma_x}{T_x}\right)^2 - \left(\frac{T_y}{T_x}\right)\left(\frac{\sigma_x}{T_x}\right)\left(\frac{\sigma_y}{T_y}\right)\right\} + \left(\frac{\tau_{xy}}{T_{xy}}\right)^2; \quad (\gamma \leq 1) \quad (27b)$$

When $\gamma > 1$, the factor of safety is the smaller of the quantities μ_1 and μ_2 defined below, using the strength matrices (25)

$$\frac{1}{u_i^2} = \left(\frac{\sigma_x}{T_x}\right)^2 + \left(\frac{\sigma_y}{T_y}\right)^2 + \left(\frac{\tau_{xy}}{T_{xy}}\right)^2 - 2h(\gamma)\left(\frac{\sigma_x}{T_x}\right)\left(\frac{\sigma_y}{T_y}\right) \pm 2g(\gamma)\left(\frac{\tau_{xy}}{T_{xy}}\right)\left(\frac{\sigma_x}{T_x} + \frac{\sigma_y}{T^y}\right); \quad (i = 1, 2; \quad \gamma > 1) \quad (28)$$

where h and g are defined in (26) and Fig. 9.

As an example of the application, let us consider a 0°/90° ±45° boron/epoxy laminate, with proportions 40%/20%/40%. The ultimate strengths of this laminate are, typically, $T_x = 95$ ksi, $T_y = 53$ ksi and $T_{xy} = 35$ ksi, as determined through a ply-by-ply strength analysis. Since the interaction factor $\gamma = 0.73$ is less than unity, the form (27) must be used to calculate the factor of safety. Under stresses $\sigma_x = 50$ ksi, $\sigma_y = 25$ ksi and $\tau_{xy} = 28$ ksi, a factor of safety $\mu = 1.02$ is found from (27). Let us now assume it is necessary to increase the laminate shear stiffness without significantly reducing longitudinal stiffness or increasing laminate thickness. This can be accomplished by converting half (for example) of the 90° plies to ±45° plies, in which case the resulting strengths become $T_{x_T} = 100$ ksi, $T_{y_T} = 35$ ksi and $T_{xy} = 40$ ksi, and the interaction factor becomes $\gamma = 1.37$. Since γ exceeds unity, the form (28) must be used. With the same stresses $\sigma_x = 50$ ksi, $\sigma_y = 25$ ksi and $\tau_{xy} = 28$ ksi, a factor of safety $\mu = 1.04$ is found.

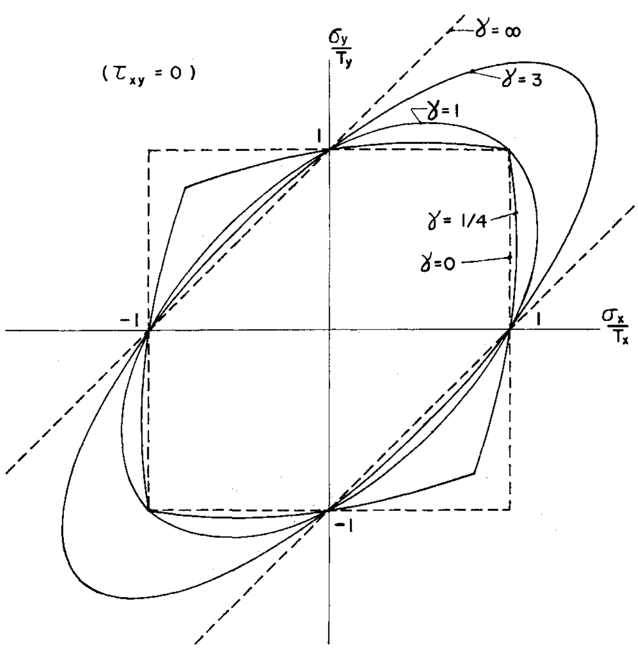

Fig. 10 Effect of the interaction factor γ on the shape of the failure surface ($\tau_{xy} = 0$).

Conclusions

A phenomenological failure theory has been presented which permits strength characterization of anisotropic materials under combined loading. When this theory is applied to laminated composites, it becomes unnecessary to recompute the stress distribution through the plies for each proposed load condition. Although the theory is generally applicable to three-dimensional stress states in anisotropic (not necessarily orthotropic) materials, it is most practically applied to orthotropic materials under states of plane stress.

This criterion can also be applied to nonlaminated anisotropic materials, such as rolled titanium, since the new concepts introduced herein, i.e., the interaction factor and the principal axes of strength, comprise a rational basis for describing the strength properties of these materials.

It has been shown that when the strength properties are defined with respect to a set of axes of orthotropy, it is not necessary to compute the principal strengths, nor is it necessary to locate a set of principal strength axes, in order to apply this criterion. The magnitude of the interaction factor γ must be considered, however, when choosing the appropriate form of the strength matrices. Since the strength matrices are actually tensors, they may be transformed in a manner analogous to that in which the matrices of the elastic constants of anisotropic materials are transformed. This also means that symmetries which affect the relationships between the elastic constants likewise affect the relationships between the strength matrix components. The major difference is that there are three (or two, for plane stress) strength matrices to be considered.

This failure criterion is well adapted to visualization, and it lends itself readily to determination of failure surfaces and safety factors. The required effort is only slightly greater than that required to apply the von Mises criterion to isotropic materials. As with all failure criteria of this type, the results are intended to predict the onset, but not the mode, of failure. Full evaluation of this, or any other, phenomenological failure criterion can be made only after a statistically significant body of experimental data has been generated.

Finally, it must be noted that, when sufficient experimental data has become available, an adjustment of this theory can be easily performed. The interaction factor γ, which has been defined as $\gamma = 3T_{xy}^2/T_xT_y$, could be defined as

$$\gamma = (3T_{xy}^2/T_xT_y)^n \tag{29}$$

without affecting the essential features of this theory. The exponent n, which could be any number greater or less than unity, might depend upon the type of material being considered. It seems reasonable to expect that a homogeneous anisotropic material such as rolled titanium might have a value n different from that of a laminated composite material.

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